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A solution differentiability result for evolutionary quasi-variational inequalities

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Abstract We consider a class of evolutionary quasi-variational inequalities arising in the study of some network equilibrium problems. First we prove the existence and uniqueness of solutions and, subsequently, present a differentiability result based on projection arguments.

Keywords Evolutionary quasi-variational inequalities · Metric projections · Network models

1 Introduction

We are concerned with existence and solution differentiability issues for a class of infinite-dimensional quasi-variational inequalities which describe network equilibrium problems in several different fields, ranging from financial markets [19] to transportation networks [2,18] (see also [5,10] for a discussion on the variational inequality formulation of some equilibrium problems). In particular, we prove that the solution to the following abstract problem belongs to $H^1(0, T; \mathbb{R}^n)$:

Find $x(t) \in K(x)$ which satisfies

$$\int_0^T \langle F(t, x(t)), y(t) - x(t)\rangle \, \mathrm{d}t \geq 0, \quad \forall y(t) \in K(x), \tag{1}$$

where $F\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a point-to-point map, $K\colon E \to 2^{L^2([0,T];\mathbb{R}^n_+)}$ is a point-to-set map with closed and convex values, E is a nonempty, compact, and convex subset of $L^2([0, T]; \mathbb{R}^n_+)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

There is a vast literature on the theory of quasi-variational inequalities. It will be beyond the scope of this paper to give a survey of all relevant results. However, we address the interested reader to the comprehensive monograph [1] and references therein. For a discussion

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of various conditions ensuring nonemptiness of the solution set of some evolutionary quasi-variational inequality problems we refer to [4,18,19].

Solution regularity has always raised the attention of numerous scholars, see for example the seminal works [6,16] in the context of nonlinear programming and [7,8,20] in the framework of variational inequalities. Recently, there has been a sharp increase in interest in sensitivity analysis and solution differentiability for variational inequality problems, as also confirmed by the numerous results in the literature, see for instance the monograph [11] and the papers [9,14,15,17,21,23]. Thus, regularity properties appear as central in applications and, in particular, in equilibrium problems. In fact, they allow us to have a complete knowledge of the solution behavior during the time horizon, and hence to describe exhaustively all the features of the model. Nevertheless, from this point of view, quasi-variational inequalities do not have an extensive literature. This is one of the motivations for the present research.

We aim at advancing the understanding of solution behavior in two directions. First, we explicitly take into account the dependence on time of the constraint set. In most of evolutionary models constraint sets are not considered as time-varying, this means that we are able to study the evolution in time of solutions, but we do not have a global time-specific description of the constraint sets. Our aim is to fill this gap by dealing with a time-dependent constraint set, i.e., $K(x) = K(x(t))$. Thus, we extend the approach examined in [9] for a variational inequality problem and show how the structure of the convex set $K(x(t))$ plays a central role and allows us to achieve fundamental solution properties. Our results were inspired by [21–23] where regularity properties of solutions to some parameterized variational inequality problems are studied. In [21] continuity and differentiability of solutions are discussed for variational inequalities under operator perturbation; in [22] a time-dependent convex set characterized by a zero obstacle is considered and regularity properties are derived for a variational inequality problem with integral term; in [23] a Hölder stability result is proved for variational inequalities with pseudo-Lipschitz parameter-dependent sets. In view of practical applications, we present a general parameter-dependent constraint set which is able encompass numerous real-life problems, ranging from transportation to economics and finance.

Second, we suggest to exploit geometric properties of constraint sets $K(x(t))$ and prove that projection arguments can be fruitfully applied to the end of proving the existence of solution derivatives and estimating their norms. Moreover, it worth noting that we obtain our result under mild and reasonable assumptions which often appear in equilibrium model descriptions. Our regularity result is of general and independent interest and it makes it possible to understand and predict choice adjustment processes of users. Moreover, it applies to all network-based models which can be cast in the form of Problem 2 (see next section) and hence has a large spectrum of possible applications.

The paper is organized as follows. In Sect. 2 the theoretical framework is presented and some basic arguments are given. Section 3 provides an existence and uniqueness result for infinite-dimensional quasi-variational inequalities. Section 4 is devoted to our main theorem on solution differentiability and, finally, Sect. 5 draws conclusions and suggests some further research issues.

2 Notations and preliminaries

Before introducing the formulation of our problem we specify our notations. For technical reasons we choose as our functional setting the Hilbert space $L^2([0, T]; \mathbb{R}^n)$, $T > 0$, of square-integrable functions from the closed interval $[0, T]$ to $\mathbb{R}^n$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{L^2} = \int_0^T \langle \cdot, \cdot \rangle \mathrm{d}t$ and the usual associated norm $\| \cdot \|_{L^2}$. The scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $\| \cdot \|$. We adopt the usual notation $H^1(0, T; \mathbb{R}^n)$ for the space of absolutely continuous functions $y\colon (0, T) \to \mathbb{R}^n$ with $\frac{\mathrm{d}y}{\mathrm{d}t} \in L^2(0, T; \mathbb{R}^n)$. Moreover, we denote by $P_A(\cdot)\colon \mathbb{R}^n \to A$ the projection operator for any closed and convex subset $A \subset \mathbb{R}^n$.

We now introduce the problem of our interest:

$$\text{Find } x(t) \in K(x)\colon \quad \langle F(t, x(t)), y(t) - x(t) \rangle_{L^2} \geq 0, \quad \forall y(t) \in K(x), \tag{2}$$

where

- $K\colon E \to 2^{L^2([0,T];\mathbb{R}^n_+)}$, with E nonempty, compact, and convex subset of $L^2([0, T]; \mathbb{R}^n_+)$, is a point-to-set map defined by

$$\begin{aligned} K(x) = \Big\{ & y \in L^2([0, T]; \mathbb{R}^n)\colon \underline{y}_i(t) \leq y_i(t) \leq \overline{y}_i(t) \text{ a.e. } t \in [0, T], \\ & i = 1, \ldots, n; \sum_{i=1}^n \xi_{ij} y_i(t) = d_j(t, x(t)) \text{ a.e. } t \in [0, T], \\ & \xi_{ij} \in \{0, 1\}, i = 1, \ldots, n; j = 1, \ldots, l \Big\}; \end{aligned} \tag{3}$$

- $\underline{y}(t), \overline{y}(t) \in L^2([0, T]; \mathbb{R}^n), 0 \leq \underline{y}(t) \leq \overline{y}(t)$;
- $d\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^l$ and $F\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$.

In order to ensure the nonemptiness of the constraint set $K(x)$, we also assume that $\Xi \underline{y}(t) \leq d(t, x(t)) \leq \Xi \overline{y}(t)$ a.e in $[0, T]$, where Ξ is the matrix with typical entry ξ_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, l$. General formulation (3) comprehends some common quasi-equilibrium problems. For instance, if $\xi_{ij} \in \{0, 1\}$ and $\underline{y}(t) \geq 0$ the set (3) represents the constraint set of traffic equilibrium problems with congestion-dependent travel demands (see [18]) and if $\xi_{ij} \in \{0, 1\}$, $\overline{y}(t)$ large enough and $\underline{y}(t) = 0$ it describes the financial equilibrium problem with implicit budget constraints (see [19]).

Remark 1 We observe that problem (2) is equivalent to the following one (see [5,10])

$$\text{Find } x(t) \in K(x)\colon \quad \langle F(t, x(t)), y(t) - x(t) \rangle \geq 0, \quad \forall y(t) \in K(x) \text{ a.e. in } [0, T]. \tag{4}$$

Now, let us recall a result which will be useful for our purposes. Let p' be the conjugate of p, let $C_c^\infty(I)$ be the set of continuous functions with compact support in the interval I and infinitely differentiable, and let $\complement A$ denote the complement of any set A.

Theorem 1 (See [3]) *Let I be an open subset of $\mathbb{R}$. If $y \in L^p(I)$, with $1 < p < \infty$, the following properties are equivalent:*

1. $y \in W^{1,p}(I)$;
2. *there exists $\gamma > 0$ such that*

$$\left| \int_I y\phi' \right| \leq \gamma \|\phi'\|_{L^{p'}(I)}, \quad \forall \phi \in C_c^\infty(I);$$

3. *there exists $\gamma > 0$ such that for any open set $\omega \subset\subset I$ and $h \in \mathbb{R}$ with $|h| < dist\colon (\omega, \complement I)$*

$$\left\| \frac{y(t+h) - y(t)}{h} \right\|_{L^p(\omega)} \leq \gamma.$$

Moreover it is possible to choose $\gamma = \|y'\|_{L^p(I)}$.

3 Existence of solutions

On the lines of Theorem 9 in [13] and Theorem 6 in [12], we claim the following statement.

Theorem 2 *Let the following assumptions be satisfied:*

(i) $F(t, y)$ *and* $d(t, y)$ *are measurable in* $t \forall y \in \mathbb{R}^n_+$, *continuous at* y *for* t *a.e. in* $[0, T]$ *and there exist* $\phi, \psi \in L^2(0, T)$ *such that*

$$\|F(t, y)\| \le \phi(t) + \|y\|, \quad \|d(t, y)\| \le \psi(t) + \|y\|;$$

(ii) $F(t, y)$ *is strongly monotone in* y, *i.e., there exists* $\alpha > 0$ *such that for* t *a.e. in* $[0, T]$

$$\langle F(t, y_1) - F(t, y_2), y_1 - y_2 \rangle \ge \alpha \|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n_+;$$

(iii) $F(t, y)$ *is Lipschitz continuous at* y, *i.e., there exists* $\beta > 0$ *such that for* t *a.e. in* $[0, T]$

$$\|F(t, y_1) - F(t, y_2)\| \le \beta \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^n_+;$$

(iv) *there exists* κ, $0 < \kappa < 1 - \sqrt{1 - \frac{\alpha^2}{\beta^2}}$, *such that* $\forall y_1, y_2 \in \mathbb{R}^n_+$,

$$\|P_{K(y_1)}(z) - P_{K(y_2)}(z)\| \le \kappa \|y_1 - y_2\|, \quad \forall z \in \mathbb{R}^n_+.$$

Then, there exists a unique solution $x(t)$ *to quasi-variational inequality problem* (2).

Proof We first observe that under assumption (i), if $y(t) \in L^2([0, T]; \mathbb{R}^n_+)$ it results that $F(t, y(t)), d(t, y(t)) \in L^2([0, T]; \mathbb{R}^n_+)$. In addition, as F and d belong to the class of Nemytskii operators, they are also continuous in L^2.

We introduce the mapping $\Sigma\colon E \to L^2([0, T]; \mathbb{R}^n_+)$ which assigns to each $u \in E$ the unique solution $w(t)$ to the parameter-dependent variational inequality (also referred as the variational section)

$$\int_0^T \langle F(t, w(t)), y(t) - w(t) \rangle \, dt \ge 0, \quad \forall y(t) \in K(u). \tag{5}$$

It immediately follows that $\Sigma(u) = w(t) = P_{K(u)}(w(t) - \lambda F(t, w(t)))$, $\lambda > 0$, and $x(t)$ solves (2) if and only if it is a fixed point of Σ. We also introduce $\Sigma'\colon L^2([0, T]; \mathbb{R}^n_+) \to L^2([0, T]; \mathbb{R}^n_+)$ defined as $\Sigma'(v) = \Sigma(P_E(v))$, which has the same fixed points as Σ. We aim to prove that Σ' is a contraction.

Let $v_1, v_2 \in L^2([0, T]; \mathbb{R}^n_+)$ and let $w_1(t), w_2(t)$ be the unique solutions to the variational sections corresponding to the sets $K(P_E(v_1))$ and $K(P_E(v_2))$, respectively. Using (iv) and the nonexpansivity of projection operators, we have

$$\begin{aligned}
\|\Sigma'(v_1) - \Sigma'(v_2)\| &= \|w_1(t) - w_2(t)\| \\
&= \|P_{K(P_E(v_1))}(w_1(t) - \lambda F(t, w_1(t))) - P_{K(P_E(v_2))}(w_2(t) - \lambda F(t, w_2(t)))\| \\
&= \|P_{K(P_E(v_1))}(w_1(t) - \lambda F(t, w_1(t))) - P_{K(P_E(v_1))}(w_2(t) - \lambda F(t, w_2(t))) \\
&\quad + P_{K(P_E(v_1))}(w_2(t) - \lambda F(t, w_2(t))) - P_{K(P_E(v_2))}(w_2(t) - \lambda F(t, w_2(t)))\| \\
&\le \|w_1(t) - w_2(t) - \lambda(F(t, w_1(t)) - F(t, w_2(t)))\| + \kappa \|P_E(v_2) - P_E(v_1)\|.
\end{aligned}$$

Moreover, by (ii) and (iii) we find

$$\|w_1(t) - w_2(t) - \lambda(F(t, w_1(t)) - F(t, w_2(t)))\|^2 \le \left(1 + \lambda^2\beta^2 - 2\alpha\lambda\right) \|w_1(t) - w_2(t)\|^2.$$

Therefore,

$$\|\Sigma'(v_1) - \Sigma'(v_2)\|\left(1 - \sqrt{1 + \lambda^2\beta^2 - 2\alpha\lambda}\right) \le \kappa\|v_1 - v_2\|$$

and choosing $\lambda = \alpha/\beta^2$ and $\kappa < 1 - \sqrt{1 - \frac{\alpha^2}{\beta^2}}$, we obtain

$$\|\Sigma'(v_1) - \Sigma'(v_2)\| \le \delta\|v_1 - v_2\|$$

with $\delta = \frac{\kappa}{1-\sqrt{1-\frac{\alpha^2}{\beta^2}}} < 1$. Hence Σ' is a contraction and its unique fixed point is also the unique solution to (2).

4 Solution differentiability

In this section, we prove the existence of solution derivatives to the following *time-dependent* problem

$$\text{Find } x(t) \in K(x(t))\colon \quad \langle F(t, x(t)), y(t) - x(t)\rangle \ge 0, \quad \forall y(t) \in K(x(t)) \tag{6}$$

with

$$\begin{aligned} K(x(t)) = \Big\{ & y \in \mathbb{R}^n\colon \underline{y}_i(t) \le y_i(t) \le \overline{y}_i(t), \\ & i = 1, \ldots, n;\ \sum_{i=1}^{n} \xi_{ij} y_i(t) = d_j(t, x(t)), \\ & \xi_{ij} \in \{0, 1\}, i = 1, \ldots, n;\ j = 1, \ldots, l \Big\}, \quad t \in [0, T]. \end{aligned} \tag{7}$$

Hereafter, with reference to the quantities introduced in Theorem 1, we choose $I = (0, T)$, $t_0 \in I, \sigma > 0, B(\sigma) =]t_0-\sigma, t_0+\sigma[\subset I, \tau \in (0, 1)$, and $B(\tau\sigma) =]t_0-\tau\sigma, t_0+\tau\sigma[\subset (0, T)$. Finally, let $h \in \mathbb{R} - \{0\}$, $|h| < (1 - \tau)\sigma$.

Theorem 3 *Let the following assumptions hold:*

(a) *$F(t, y)$ is strongly monotone in y, i.e., there exists $\alpha > 0$ such that, $\forall t \in [0, T]$,*

$$\langle F(t, y_1) - F(t, y_2), y_1 - y_2\rangle \ge \alpha\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n;$$

(b) *$F(t, y)$ is Lipschitz continuous at y, i.e., there exists $\beta > 0$ such that, $\forall t \in [0, T]$,*

$$\|F(t, y_1) - F(t, y_2)\| \le \beta\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^n;$$

(c) *there exists $M > 0$ such that, for $t_1, t_2 \in [0, T]$,*

$$\|F(t_1, y) - F(t_2, y)\| \le M\|y\||t_1 - t_2|, \quad \forall y \in \mathbb{R}^n;$$

(d) *there exists κ satisfying $0 \le \kappa < 1 - \sqrt{1 - \frac{\alpha^2}{\beta^2}}$, such that, $\forall t_1, t_2 \in [0, T]$,*

$$\|P_{K(x(t_1))}(z) - P_{K(x(t_2))}(z)\| \le \kappa\|x(t_1) - x(t_2)\|, \quad \forall z \in \mathbb{R}^n.$$

Then, the unique solution $x(t)$ to problem (6) belongs to $H^1(0, T; \mathbb{R}^n)$ and for any open set $\omega \subset\subset (0, T)$ the following estimate holds

$$\|x'\|^2_{L^2(\omega;\mathbb{R}^n)} \le \gamma\|x\|^2_{L^2(0,T;\mathbb{R}^n)},$$

where $\gamma = \gamma(\alpha, \beta, M, dist(\omega, \complement I))$.

Proof We start with reformulating quasi-variational inequality problem (6) as a classical fixed-point problem. Therefore, we may write

$$x(t) = P_{K(x(t))}(x(t) - \lambda F(t, x(t))),$$
$$x(t+h) = P_{K(x(t+h))}(x(t+h) - \lambda F(t+h, x(t+h)))$$

with $\lambda > 0$. In order to simplify notations, we set $x(t) = x$, $x(t+h) = x_h$ and $\Delta x = x_h - x$. Thus, we have

$$\left\|\frac{\Delta x}{h}\right\|^2 = \left\|\frac{x_h - x}{h}\right\|^2 = \left\|\frac{P_{K(x_h)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x_h - \lambda F(t+h, x_h))}{h}\right.$$
$$\left. + \frac{P_{K(x)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x - \lambda F(t, x))}{h}\right\|^2$$
$$\leq \left(\left\|\frac{P_{K(x_h)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x_h - \lambda F(t+h, x_h))}{h}\right\|\right.$$
$$\left. + \left\|\frac{P_{K(x)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x - \lambda F(t, x))}{h}\right\|\right)^2.$$

Using inequality $(a+b)^2 \leq (1+\eta)a^2 + (1+\frac{1}{\eta})b^2$, with $\eta > 0$ sufficiently small, hypothesis (d) and the nonexpansivity of projections, we continue the inequality chain as follows

$$\left\|\frac{\Delta x}{h}\right\|^2 \leq (1+\eta)\left\|\frac{P_{K(x_h)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x_h - \lambda F(t+h, x_h))}{h}\right\|^2$$
$$+\left(1+\frac{1}{\eta}\right)\left\|\frac{P_{K(x)}(x_h - \lambda F(t+h, x_h)) - P_{K(x)}(x - \lambda F(t, x))}{h}\right\|^2$$
$$\leq (1+\eta)\kappa^2\left\|\frac{\Delta x}{h}\right\|^2 + \left(1+\frac{1}{\eta}\right)\left\|\frac{\Delta x}{h} - \lambda\frac{F(t+h, x_h) - F(t, x)}{h}\right\|^2. \quad (8)$$

We now estimate the squared norm appearing in (8)

$$\left\|\frac{\Delta x}{h} - \lambda\frac{F(t+h, x_h) - F(t, x)}{h}\right\|^2$$
$$= \left\|\frac{\Delta x}{h} - \lambda\frac{F(t+h, x_h) - F(t+h, x)}{h} - \lambda\frac{F(t+h, x) - F(t, x)}{h}\right\|^2$$
$$= \left\|\frac{\Delta x}{h} - \lambda\frac{F(t+h, x_h) - F(t+h, x)}{h}\right\|^2 + \lambda^2\left\|\frac{F(t+h, x) - F(t, x)}{h}\right\|^2$$
$$-2\lambda\langle\frac{\Delta x}{h}, \lambda\frac{F(t+h, x) - F(t, x)}{h}\rangle$$
$$+2\lambda^2\langle\frac{F(t+h, x_h) - F(t+h, x)}{h}, \frac{F(t+h, x) - F(t, x)}{h}\rangle$$
$$\leq \left\|\frac{\Delta x}{h}\right\|^2 + \lambda^2\left\|\frac{F(t+h, x_h) - F(t+h, x)}{h}\right\|^2$$
$$-2\lambda\langle\frac{\Delta x}{h}, \frac{F(t+h, x_h) - F(t+h, x)}{h}\rangle + \lambda^2\left\|\frac{F(t+h, x) - F(t, x)}{h}\right\|^2$$
$$+2\lambda\left\|\frac{\Delta x}{h}\right\| \cdot \left\|\frac{F(t+h, x) - F(t, x)}{h}\right\|$$
$$+2\lambda^2\left\|\frac{F(t+h, x_h) - F(t+h, x)}{h}\right\| \cdot \left\|\frac{F(t+h, x) - F(t, x)}{h}\right\|.$$

Now from (a)–(c) and using twice inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$, with $\epsilon = \epsilon_1, \epsilon_2 > 0$ sufficiently small, we continue the above inequality chain

$$\begin{aligned}
&\left\| \frac{\Delta x}{h} - \lambda \frac{F(t+h, x_h) - F(t,x)}{h} \right\|^2 \\
&\le \left\| \frac{\Delta x}{h} \right\|^2 \left(1 + \lambda^2\beta^2 - 2\alpha\lambda\right) + \lambda^2 M^2 \|x\|^2 \\
&\quad + 2\lambda\left(\epsilon_1 \left\| \frac{\Delta x}{h}\right\|^2 + \frac{1}{4\epsilon_1}\left\| \frac{F(t+h,x) - F(t,x)}{h}\right\|^2\right) \\
&\quad + 2\lambda^2\left(\epsilon_2 \left\| \frac{F(t+h,x_h) - F(t+h,x)}{h}\right\|^2 + \frac{1}{4\epsilon_2}\left\| \frac{F(t+h,x) - F(t,x)}{h}\right\|^2\right) \\
&\le \left\| \frac{\Delta x}{h} \right\|^2 \left(1 + \lambda^2\beta^2 - 2\alpha\lambda\right) + 2\left\| \frac{\Delta x}{h}\right\|^2 \lambda(\epsilon_1 + \lambda\beta^2\epsilon_2) \\
&\quad + \|x\|^2 M^2 \lambda\left(\lambda + \frac{1}{2\epsilon_1} + \frac{\lambda}{2\epsilon_2}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left\| \frac{\Delta x}{h} \right\|^2 \le \left\| \frac{\Delta x}{h} \right\|^2 &\left((1+\eta)\kappa^2 + \left(1 + \frac{1}{\eta}\right)\left(1 + \lambda^2\beta^2 - 2\alpha\lambda\right)\right. \\
&\left. + 2\left(1 + \frac{1}{\eta}\right)\lambda(\epsilon_1 + \lambda\beta^2\epsilon_2)\right) \\
& + \|x\|^2 \left(1 + \frac{1}{\eta}\right) M^2 \lambda\left(\lambda + \frac{1}{2\epsilon_1} + \frac{\lambda}{2\epsilon_2}\right).
\end{aligned}$$

In conclusion, we get

$$\begin{aligned}
&\left\| \frac{\Delta x}{h} \right\|^2 \left(1 - \left((1+\eta)\kappa^2 + \left(1 + \frac{1}{\eta}\right)\left(1 + \lambda^2\beta^2 - 2\alpha\lambda\right)\right) - 2\left(1 + \frac{1}{\eta}\right)\lambda(\epsilon_1 + \lambda\beta^2\epsilon_2)\right) \\
&\le \|x\|^2 \left(1 + \frac{1}{\eta}\right) M^2 \lambda\left(\lambda + \frac{1}{2\epsilon_1} + \frac{\lambda}{2\epsilon_2}\right),
\end{aligned}$$

and, choosing $\lambda = \alpha/\beta^2$, we obtain

$$\begin{aligned}
&\left\| \frac{\Delta x}{h} \right\|^2 \left(1 - \left((1+\eta)\kappa^2 + \left(1 + \frac{1}{\eta}\right)\left(1 - \frac{\alpha^2}{\beta^2}\right)\right) - 2\left(1 + \frac{1}{\eta}\right)\frac{\alpha}{\beta^2}\left(\epsilon_1 + \alpha\epsilon_2\right)\right) \\
&\le \|x\|^2 \left(1 + \frac{1}{\eta}\right) M^2 \frac{\alpha}{\beta^2}\left(\frac{\alpha}{\beta^2} + \frac{1}{2\epsilon_1} + \frac{\alpha}{2\epsilon_2\beta^2}\right).
\end{aligned}$$

Now, we observe that

$$1 - \left((1+\eta)\kappa^2 + \left(1 + \frac{1}{\eta}\right)\left(1 - \frac{\alpha^2}{\beta^2}\right)\right) > 0,$$

if

$$\eta \in \left] - \frac{(\kappa^2 - \frac{\alpha^2}{\beta^2}) + \sqrt{(\kappa^2 + \frac{\alpha^2}{\beta^2})^2 - 4\kappa^2}}{2\kappa^2}, - \frac{(\kappa^2 - \frac{\alpha^2}{\beta^2}) - \sqrt{(\kappa^2 + \frac{\alpha^2}{\beta^2})^2 - 4\kappa^2}}{2\kappa^2} \right[. \quad (9)$$

Thus, for sufficiently small values of ϵ_1, ϵ_2 and under condition (9), it results that

$$c = \left(1 - \left((1+\eta)\kappa^2 + \left(1 + \frac{1}{\eta}\right)\left(1 - \frac{\alpha^2}{\beta^2}\right)\right) - 2\left(1 + \frac{1}{\eta}\right)\frac{\alpha}{\beta^2}\left(\epsilon_1 + \alpha\epsilon_2\right)\right) > 0.$$

Hence we obtain

$$\left\| \frac{\Delta x}{h} \right\|^2 \leq \overline{c} \|x\|^2 \tag{10}$$

with

$$\overline{c} = c^{-1}\Big(1 + \frac{1}{\eta}\Big) M^2 \frac{\alpha}{\beta^2} \Big(\frac{\alpha}{\beta^2} + \frac{1}{2\epsilon_1} + \frac{\alpha}{2\epsilon_2 \beta^2} \Big).$$

Integrating between $t_0 - \tau\sigma$ and $t_0 + \tau\sigma$, it results that

$$\left\| \frac{\Delta x}{h} \right\|^2_{L^2(B(\tau\sigma);\mathbb{R}^n)} \leq \overline{c} \|x\|^2_{L^2(B(\tau\sigma);\mathbb{R}^n)}.$$

Thus, we may apply Theorem 1 with $\gamma = \overline{c}\|x\|^2_{L^2(0,T;\mathbb{R}^n)}$ and conclude that $x \in H^1(B(\sigma);\mathbb{R}^n)$. As a consequence, $x \in H^1(0,T;\mathbb{R}^n)$ and for any open set $\omega \subset\subset (0,T)$ the estimate

$$\|x'\|^2_{L^2(\omega;\mathbb{R}^n)} \leq \gamma \|x\|^2_{L^2(0,T;\mathbb{R}^n)}$$

holds.

Remark 2 Using arguments similar to those applied in [9], it is possible to estimate the variation rate of projections appearing in assumption (d). Specifically, if map $d(t, y(t))$ is Lipschiz continuous with respect to y with Lipschitz constant L, a lower bound for the quantity κ which depends on constant L can be obtained.

Theorem 3 entitles us to reformulate variational inequality problem (4) as the time-dependent problem (6), and hence, by means of Remark 1, the regularity result holds true for the initial problem (2).

5 Conclusions

In this paper, we focused on a class of evolutionary quasi-variational inequalities arising in the study of equilibrium problems in network-based models. By means of projection arguments we were able not only to prove the existence and uniqueness of solutions, but also a differentiability result as well as some bounds for the norm of the derivative.

Regularity properties have a large spectrum of possible applications as they ensure a better understanding of solution behavior and make it possible to predict changes during the time horizon.

Future extensions of the work include the following issues. First, a Volterra integral term can be introduced in the model so as to express the influence of the past flow distribution through the network (see [9]). Second, the quasi-variational inequality problem can be reformulated under a different structure of the constraint set and thus adapted to a different network equilibrium framework.

References

1. Baiocchi, F., Capelo, A.: Variational and Quasivariational Inequalities. Applications to free boundary problems. Wiley, New York (1984)
2. Bliemer, M., Bovy, P.: Quasi-variational inequality formulation of the multiclass dynamic traffic assignment problem. Transport. Res. B **37**, 501–519 (2003)
3. Brezis, H.: Analyse fonctionelle. Théorie et applications. Masson, New York (1983)

4. Cubiotti, P.: Generalized quasi-variational inequalities in infinite-dimensional normed spaces. J. Optim. Theory Appl. **92**(3), 457–475 (1999)
5. Daniele, P.: Dynamic networks and evolutionary variational inequalities. Edward Elgar Publishing (2006)
6. Fiacco, A.V., McCormick, G.P.: Nonlinear programming: sequential unconstrained minimization techniques. Wiley, New York (1968)
7. Kyparisis, J.: Uniqueness and differentiability of solutions of parametric nonlinear complementarity problems. Math. Program. **36**, 105–113 (1986)
8. Kyparisis, J.: Solution differentiability for variational inequalities. Math. Program. **48**, 285–301 (1990)
9. Maugeri, A., Scrimali, L.: On the differentiability of solutions to variational inequalities with long-term memory. to appear
10. Maugeri, A., Vitanza, C.: Time-dependent equilibrium problems. In: Migdalos, A., Pardalos, P., Pitsoulis, L. (eds.) Pareto Optimality Game Theory and Equilibria, Nonconvex Optimization and its Applications Series. Springer, New York (2005)
11. Mordukhovich, B.: Variational analysis and generalized differentiation. Springer, New York (2006)
12. Nesterov, Yu., Scrimali, L.: Solving strongly monotone variational and quasi-variational inequalities. CORE Discussion Papers (2006/107)
13. Noor, M.A., Oettli, W.: On general nonlinear complementarity problems and quasiequilibria. Le Matematiche **XLIX**(II), 313–331 (1994)
14. Robinson, S.M.: Some continuity properties of polyhedral multifunctions. Math. Program. Study **1**, 206–214 (1981)
15. Robinson, S.M.: Generalized equations and their solutions, part II: applications to nonlinear programming. Math. Program. Study **19**, 200–221 (1982)
16. Robinson, S.M.: Implicit B-differentiability in generalized equations, Technical Summary Report 2854. Mathematics Research center, University of Wisconsin, Madison, WI (1985)
17. Robinson, S.M., Lu, S.: Solution continuity in variational conditions. J. Glob. Optim. to appear
18. Scrimali, L.: Quasi-variational inequalities in transportation networks. Math. Models Methods Appl. Sci. **14**(10), 1541–1560 (2004)
19. Scrimali, L.: The financial equilibrium problem with implicit budget constraints. to appear
20. Shapiro, A.: Sensitivity analysis of nonlinear programs and differentiability properties of metric projections. SIAM J. Control Optim. **26**, 628–645 (1988)
21. Shapiro, A.: Sensitivity analysis of parameterized variational inequalities. Math. Oper. Res. **30**(1), 109–126 (2005)
22. Steinbach, J.: On a variational inequality containing a memory term with an application in electro-chemical machining. J. Convex Anal. **5**(1), 63–80 (1998)
23. Yen, N.D.: Hölder continuity of solutions to a parametric variational inequality. Appl. Math. Optim. **31**, 245–255 (1995)